ON HOMOGENEOUS TERNARY QUADRATIC DIOPHANTINE EQUATION $z^2 = 47x^2 + y^2$

R. Anbuselvi [1] | N. Ahila [2]

[1] Department of Mathematics, A.D.M. College for Women (Autonomous), Nagapattinam – 600 001, Tamil Nadu, India.

[2] Department of Mathematics, Thiru.Vi.Ka. Govt. Arts College, Tiruvarur- 610003, Tamil Nadu, India.

ABSTRACT

The ternary quadratic homogeneous equation representing homogenous cone given by $z^2 = 47x^2 + y^2$ is analyzed for its non-zero distinct integer points on it. Four different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number, Octahedral number, Stella Octangular number and Oblong number are presented. Also knowing an integer solution satisfying the given cone, two triples of integers generated from the given solution are exhibited.

KEYWORDS: Ternary homogeneous quadratic, integral solutions.

Introduction

The Ternary quadratic Diophantine equations offer an unlimited field for research because of their variety [1-5]. For an extensive view of various problem one may refer [6-20]. This communication concerns with yet another interesting for ternary quadratic equation $z^2 = 47x^2 + y^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

Notations used

- $T_{m,n}$ - Polygonal number of rank n with size m.
- CP_n^m - Centered Pyramidal number of rank n with size m.
- Pr_n - Pronic number of rank n.
- SO_n - Stella Octangular number of rank n.
- Obl_n - Oblong number of rank n.
- OH_n - Octahedral number of rank n.
- Pt_n - Pentatope number of rank n.
- PP_n - Pentagonal Pyramidal number of rank n

Method of analysis

The ternary quadratic equation under consideration is

$$z^2 = 47x^2 + y^2 \tag{1}$$

we have different pattern of solutions of (1) which are illustrated below.

Pattern-I

Consider (1) as

$$z^2 * 1 = 47x^2 + y^2 \tag{2}$$

Assume

$$z = a^2 + 47b^2 \tag{3}$$

Write 1 as

$$1=\frac{\left\{\left[\left(23+2n-2n^{2}\right)+i\sqrt{47}\left(2n-1\right)\right]\left[\left(23+2n-2n^{2}\right)-i\sqrt{47}\left(2n-1\right)\right]\right\}}{\left(24-2n+2n^{2}\right)^{2}}$$

Substituting (3) and (4) in (2) and employing the method of factorization, define

$$y+i\sqrt{47}\,x=\frac{\left\{\left[\left(23+2n-2n^{2}\right)+i\sqrt{47}\left(2n-1\right)\right]\left(a+i\sqrt{47}\,b\right)^{2}\right\}}{\left(24-2n+2n^{2}\right)}$$

Equating the real and imaginary parts in the above equation, we get

$$x=\frac{\left[\left(23+2n-2n^{2}\right)2ab+\left(2n-1\right)\left(a^{2}-47b^{2}\right)\right]}{\left(24-2n+2n^{2}\right)}$$

$$y=\frac{\left[\left(23+2n-2n^{2}\right)\left(a^{2}-47b^{2}\right)+\left(2n-1\right)94ab\right]}{\left(24-2n+2n^{2}\right)}$$

Replacing a by ($24-2n+2n^2$) A, b by($24-2n+2n^2$)B in the above equation the corresponding integer solutions of (1) are given by

$$x=\left(24-2n+2n^{2}\right)\left[\left(23+2n-2n^{2}\right)2AB+\left(2n-1\right)\left(A^{2}-47B^{2}\right)\right]$$

$$y=\left(24-2n+2n^{2}\right)\left[\left(23+2n-2n^{2}\right)\left(A^{2}-47B^{2}\right)-30AB\left(2n-1\right)\right]$$

$$z=\left(24-2n+2n^{2}\right)^{2}\left[A^{2}+47B^{2}\right]$$

For simplicity and clear understanding, taking n=1 in the above equations, the corresponding integer solutions of (1) are given by

$$x=24A^{2}-1128B^{2}+1104AB$$
$$y=552A^{2}-25944B^{2}-2256AB$$
$$z=576A^{2}+27072B^{2}$$

Properties

1) $24x\left(A,7A^{2}-4\right)-z\left(A,7A^{2}-4\right)-79488CP_{A}^{14}-1152\Pr_{A}\equiv 0\pmod{1152}$

2) $x\left(B+1,B\right)+y\left(B+1,B\right)-z\left(B+1,B\right)+1152\Pr_{B}+T_{108290,B}\equiv 0\pmod{54143}$

3) $x\left(A,2A^{2}-1\right)+y\left(A,2A^{2}-1\right)+z\left(A,2A^{2}-1\right)-1152obl_{A}+1152SO_{A}\equiv 0\pmod{1152}$

4) $23x\left(A,A\left(A+1\right)\right)-y\left(A,A\left(A+1\right)\right)-55296PP_{A}\equiv 0$

5) $x\left(A,1\right)-1128obl_{A}+T_{2210,A}\equiv 1128\pmod{1127}$

6) $z\left(1,B\right)-27072\Pr_{B}\equiv 576\pmod{27072}$

Pattern-II

It is worth to note that 1 in (2) may also be represented as

$$1=\frac{\left\{\left[\left(47-4n^2\right)+i\sqrt{47}\,(4n)\right]\left[\left(47-4n^2\right)-i\sqrt{47}\,(4n)\right]\right\}}{\left(47+4n^2\right)^2}$$

Following the analysis presented above, the corresponding integer solution to (1) are found to be

$$x=\left(47+4n^2\right)\left[\left(47-4n^2\right)2AB+4n\left(A^2-47B^2\right)\right]$$

$$y=\left(47+4n^2\right)\left[\left(47-4n^2\right)\left(A^2-47B^2\right)-376ABn\right]$$

$$z=\left(47+4n^2\right)^2\left[A^2+47B^2\right]$$

For the sake of simplicity, taking n=1 in the above equations, the corresponding integer solutions of (1) are given by

$$x=204\,A^2-9588\,B^2+4386\,AB$$

$$y=2193A^2-103071B^2-19176AB$$

$$z=2601\,A^2+122247\,B^2$$

Properties

1) $43x\left(A,A(A+1)\right)-4y\left(A,A(A+1)\right)-191126PP_A\equiv 0$

2) $2193x\left(A,2A^2-1\right)-204y\left(A,2A^2-1\right)-13530402SO_A\equiv 0$

3) $2193x\left(A,2A^2+1\right)-204y\left(A,2A^2+1\right)-40591206OH_A\equiv 0$

4) $x(A,1)-204\Pr_A\equiv -9588(\mathrm{mod}\,4182)$

5) $43x\left(A,(A+1)(A+2)(A+3)\right)-4y\left(A,(A+1)(A+2)(A+3)\right)-2293512\,Pt_A\equiv 0$

Pattern-III

Equation (1) can be written as

$$\frac{z+y}{47x}=\frac{x}{z-y}=\frac{P}{Q} \qquad (5)$$

From equation (5), we get two equations

$$47Px-Qy-Qz=0$$

$$Qx+Py-Pz=0$$

We get the integer solutions are

$$x=x(P,Q)=2PQ$$

$$y = y(P,Q) = 47P^2 - Q^2$$

$$z = z(P,Q) = 47P^2 + Q^2$$

Properties

1) $y(a,a) + z(a,a) - 94\Pr_a \equiv 0(\bmod 94)$

2) $x(a,-a^2) - 2CP_a^6 \equiv 0$

3) $x(P,\ 2P^2-1) + y(P,2P^2-1) + z(P,2P^2-1) - 218\Pr_P - T_{250,P} + 2SO_P \equiv 0(\bmod 341)$

4) $x(P,1) + y(P,1) - 47obl_P \equiv -1(\bmod 49)$

2. Generation of integer solutions

Let (x_0, y_0, z_0) be any given integer solution of (1). Then, each of the following triples of integers satisfies (1):

Triple 1: (x_1, y_1, z_1)

$$x_1 = 5^n x_0$$

$$y_1 = \frac{1}{10}\left(\left(18(5)^n - 8(-5)^n\right)y_0 + \left(12(5)^n - 12(-5)^n\right)z_0\right)$$

$$z_1 = \frac{1}{10}\left(\left(12(-5)^n - 12(5)^n\right)y_0 + \left(18(-5)^n - 8(5)^n\right)z_0\right)$$

Triple 2:(x_2, y_2, z_2)

$$x_2 = \frac{1}{4}\left(\left(98(2)^n - 94(-2)^n\right)x_0 + \left(14(-2)^n - 14(2)^n\right)z_0\right)$$

$$y_2 = 2^n y_0$$

$$z_2 = \frac{1}{4}\left(\left(658(2)^n - 658(-2)^n\right)x_0 + \left(98(-2)^n - 94(2)^n\right)z_0\right)$$

Conclusion

In this paper, we have presented four different patterns of infinitely many non-zero distinct integer solution of the homogeneous cone given by $z^2 = 47x^2 + y^2$. To conclude, one may search for other patterns of solutions and their corresponding properties.

References

[1] Dickson, L.E. “History of Theory of Numbers and Diophantine Analysis”, Vol.2, Dover Publications, New York 2005.

[2] Mordell L.J., “Diophantine Equations” Academic Press, new York, 1970

[3] RD. Carmicheal, “The Theory of Numbers and Diophantine Analysis”, Dover Publications, New York 1959

[4] M.A. Gopalan, ManjuSomnath and N.Vanitha, “On ternary cubic Diophantine equation x2+y2 = 2z3 Advances in Theoretical and Applied Mathematics”, Vol. 1, no.3, 227-231, 2006.

[5] M.A. Gopalan, ManjuSomnath and N. Vanitha, “Ternary cubic Diophantine equation x2 – y2 = z3”, ActaCienciaIndica”, Vol.XXXIIIM, No. 3, 705-707, 2007.

[6] M.A. Gopalan and R.Anbuselvi, “Integral Solutions of ternary cubic Diophantine equation x2 + y2 + 4N = zxy”, Pure and Applied Mathematics Sciences Vol.LXVII, No. 1-2, 107-111, March 2008.

[7] M.A. Gopalan, ManjuSomnath and N. Vanitha, “Ternary cubic Diophantine equation 22a-1 (x^2+y^2) = z^3, actaCienciaIndia”, Vol.XXXIVM, No. 3, 1135-1137, 2008.

[8] M.A. Gopalan, J.Kaligarani Integral solutions $x^3 + y^3 + 8k(x+y) = 2k+1)z^3$ Bulletin of Pure and Applied Sciences, vol 29E, No.1, 95-99,2010

[9] M.A.Gopalan, J.Kaligarani Integral solutions x3 + y3 + 8k(x+y) = (2k+1)z3 Bulletin of Pure and Applied Sciences, Vol. 29E, No.1, 95-99, 2010

[10] M.A.Gopalan, S.Premalatha On the ternary cubic equation $x^3 + x^2 + y^3 – y^2 = 4(z^3 + z^2)$ Cauvery Research Journal Vol. 4, iss 1&2 87-89, July 2010-2011.

[11] M.A.Gopalan, V.Pandichelvi, observations on the ternary cubic diophantine equation $x^3 + y^3 + x^2 - y^3= 4(z^3 + z^2)$ Archimedes J.Math 1(1), 31-37, 2011

[12] M.A.Gopalan, G.Srividhya Integral solutions of ternary cubic diophantine equation $x^3 + y^3 = z^2$ ActaCienciaIndica, Vol XXXVII No.4, 805-808, 2011

[13] M.A.Gopalan, A.Vijayashankar, S.Vidhyalakshmi Integral solutions of ternary cubic equation $x^2+ y^2– xy + 2(x + y + z) = (k^2+ 3)z^3$ Archimedes J.Math 1(1), 59-65, 2011

[14] M.A.Gopalan, G.Sangeetha on the ternary cubic diophantine equation $y^2=Dx^2+z^3$ Archimedes J.Math 1(1), 7-14, 2011

[15] M.A.Gopalan and B.Sivakami, “Integral Solutions of the ternary cubic Diophantine equation 4x2 – 4xy + 6y2 = [(k+1)2+5] w3” Impact J.Sci. Tech., vol. 6, No.1, 15-22, 2012

[16] M.A.Gopalan, S.Vidhyalakshmi and G.Sumathi, on the non-homogeneous equation with three unknowns “$x^3+y^3 = 14z^3+3(x+y)$” Discovery Science, Vol.2, No:4,37-39. Oct.2012

[17] M.A.Gopalan, B.Sivakami Integral solutions of the ternary cubic equation $4x^2 – 4xy + 6y^2 = ((k+1)2+5)w^3$ Impact J. Sci. Tech., Vol 6 No. 1, 15-22, 2012

[18] M.A.Gopalan, B.Sivakami on the ternary cubic diophantine equation $2xz = y^2(x+z)$ Bessel. J.Math 2(3), 171-177, 2012.

[19] S.Vidhyalakshmi, T.R.Usha Rani and M.A.Gopalan Integral solutions of non-homogenous ternary cubit equation $ax^2+by^2 = (a+b)z^3$DiophantusJ.Math, 2(1), 31-38, 2013

[20] M.A.Gopalan, K.Geetha On the ternary cubic diophantine equation $x^2 + y^2 – xy = z^3$ Bessel J.Math., 3(2), 119-123, 2013

[21] M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha Observations on the ternary cubic equation $x^2 + y^2 + xy = 12z^3$Antartica J.Math.10(5), 453-460, 2013

[22] M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi Lattice points on the non homogeneous ternary cubic equation $x^3 + y^3 + z^3 (x+y+z) = 0$ Impact J. Sci. Tech., Vol 7 No.1, 21-25,2013

[23] M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi Lattice points on the non homogeneous ternary cubic equation $x^3 + y^3 + z^3 (x+y+z) = 0$ Impact J. Sci. Tech., Vol 7 No.1, 51-55,2013

[24] M.A.Gopalan, S.Vidhyalakshmi and S.Mallika on the ternary non homogeneous cubic equation $x^3 + y^3 - 3 (x+y) = 2(3k^2-2)z^3$ Impact J. Sci. Tech., Vol 7 No.1, 41-55,2013